Economic interpretation of the Lagrange multiplier

Let the utility function $U(x_1, x_2)$ be subject to an income constraint: $x_1 + x_2 = I$. The Lagrangian function associated with this constrained maximization problem is: $\mathcal{L}(x_1, x_2, \lambda) = U(x_1, x_2) + \lambda(I - x_1 - x_2)$. Show that at the optimum point, the value of λ represents the marginal utility of income.

Solution

By the envelope theorem, we can calculate the derivative of the Lagrangian with respect to income and then evaluate the function at the optimal values[1], concluding that the marginal utility of income is lambda.
The result of the envelope theorem can be seen as follows: We know that at the optimum the values of x_1, x_2 and λ depend on I, so deriving the Lagrangian and using the chain rule:

$$\frac{\partial \mathcal{L}}{\partial I} = \frac{\partial \mathcal{L}}{\partial x_1} * \frac{\partial x_1}{\partial I} + \frac{\partial \mathcal{L}}{\partial x_2} * \frac{\partial x_2}{\partial I} + \frac{\partial \mathcal{L}}{\partial \lambda} * \frac{\partial \lambda}{\partial I} + \lambda$$

Now we replace x_1, x_2 and λ with their values at the optimum.

$$\frac{\partial \mathcal{L}}{\partial I} = \frac{\partial \mathcal{L}}{\partial x_1^*} * \frac{\partial x_1^*}{\partial I} + \frac{\partial \mathcal{L}}{\partial x_2^*} * \frac{\partial x_2^*}{\partial I} + \frac{\partial \mathcal{L}}{\partial \lambda^*} * \frac{\partial \lambda^*}{\partial I} + \lambda^*$$

By first order conditions:

$$\frac{\partial \mathcal{L}}{\partial x_1^*} = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2^*} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda^*} = 0$$

So we have:

$$\frac{\partial \mathcal{L}}{\partial I} = \lambda^*$$

On the other hand, we also know that at the optimum the constraint is met with equality and therefore:

$$\mathcal{L} = U + \lambda(0) = U$$

Therefore, the marginal utility of income is λ^*

$$\frac{\partial U}{\partial I} = \lambda^* \tag{1}$$

Finally, we can also say that the interpretation of λ can be made in any maximization or minimization problem. The idea is that this parameter represents how the objective function (in this case, utility) changes when we relax the constraint (in this case, it means giving more income to the individual). It is also often said to be a shadow price since this parameter indicates how much an individual would be willing to pay for an increase in income, as the value of λ tells me how much utility increases by having more income.

[1] In a nutshell, the envelope theorem tells us that if we find that the optimum of a function is x^*, then deriving that optimum with respect to a parameter is the same as deriving the original function with respect to the parameter and then replacing the result with x^*.